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COMMENT

On the number of lattice animals embeddable in the square lattice

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Abstract. Enumeration of lattice animals embeddable in a square lattice has recently been extended to 24 cell animals by Redelmeier. It is shown that the number of animals per site a_n is given by $a_n \sim 0.317(4.0626)^n n^{-1} \exp(-0.465\, n^{-0.87})$ to a high degree of accuracy.

1. Introduction

In this comment, we give some refined numerical estimates of the critical parameters in the general asymptotic form for the number of site animals per site of the square lattice. The asymptotic form is that first proposed by Domb (1976), who suggested that $a_n \sim A\lambda^n n^{-\tau} \exp(-Fn^{1-\theta})$, where a_n is the number of n-cell site animals per site, and A, λ, τ, F and θ are constants. This form was proposed by Domb in order to explain the relatively slow convergence of the sequence $\{a_n\}$ when extrapolated under the assumption that $a_n \sim A\lambda^n n^{-\tau}$ (Sykes and Glen 1976). Domb's asymptotic form was subsequently used by Guttmann and Gaunt (1978), who showed that, for all available bond and site animal series in both two and three dimensions, Domb's form appeared to fit the data better than the simpler form used by Sykes and Glen (1976), corresponding to $F = 0$.

Since that time, two important developments have taken place. Firstly, Domb's expression has received additional theoretical support (see Harris and Lubensky 1981 and references therein), and the square lattice site animal series has been extended by a full five terms (Redelmeier 1981) up to and including a_{24} in a computation that took ten *months* of CPU time on a PDP 11/70, using a highly efficient algorithm. We note in passing, that Redelmeier is concerned by a discrepancy between his value of a_{17} and that of Lunnon (1971). However, his value is confirmed by Sykes and Glen (1976), so it is clear that Lunnon's coefficient is in error.

As in most non-trivial enumeration problems, there are very few exact results and a number of bounds. The most significant exact result is due to Klarner (1967) who showed that

$$\lim_{n\to\infty} n^{-1} \ln a_n = \sup_{n>0} n^{-1} \ln a_n = \ln \lambda. \tag{1}$$

These site animals are also called *fixed polyominoes*, to distinguish them from free

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polyominoes. They differ in that *free polyominoes* are not considered distinct if they differ only in orientation. Thus ,  are two distinct fixed polyominoes (hence $a_2 = 2$), while they represent the same free polyomino (hence $p_2 = 1$), where p_n denotes the number of n-cell free polyominoes per site. Klarner (1967) was also able to show that (1) holds for free polyominoes—with p_n replacing a_n—while Lunnon (1971) showed that, for the square lattice, $a_n \sim 8p_n$. As pointed out by Whittington and Gaunt (1978) this asymptotic relation is rapidly reached, as by $n = 18$ we have $a_n/p_n = 7.99919\ldots$. Since Redelmeier also enumerated the number of free polyominoes up to and including $n = 24$, we now find $a_{24}/p_{24} = 7.999986\ldots$. Several authors have given rigorous bounds on λ. Eden (1961) gave $3.14 < \lambda < 4$, but his proof of the upper bound is false—and indeed the upper bound is not an upper bound at all. Klarner (1967) proved that $3.722 < \lambda < 6.75$, while Lunnon (1971) reports an unpublished result of Conway and Guy—which he had not seen—that allegedly established the result $\lambda < 4.5$. The best published upper bound is due to Klarner and Rivest (1973), who obtained $\lambda < 4.65$. In 1978 Whittington and Gaunt studied the general d-dimensional polyomino problem and showed that $\ln \lambda(d) \geqslant m^{-1} \ln[da_m(d)]$. Using the last coefficient obtained by Redelmeier, a_{24}, we get the bound $\lambda(2) > 3.487$, which is weaker than Klarner's (1967) result. To improve on Klarner's result using the lower bound of Whittington and Gaunt would require knowledge of a_{32}, which is computationally quite unrealistic using any known algorithm. A more feasible approach would be to sharpen the lower bound. If it could be proved that $\ln \lambda(2) \geqslant m^{-1} \ln[ka_m]$, with $k \geqslant 9.6$, then Redelmeier's last coefficient, a_{24}, would suffice to improve Klarner's lower bound. Probably the only way to improve k so substantially would be to prove the result $\ln \lambda \geqslant m^{-1} \ln[ma_m]$, which we expect to be true, though we have been unable to establish a proof, and which would give $\lambda > 3.868$.

Turning now to the asymptotic form assumed by Domb, there have been several subsequent studies of the average number of clusters in the general percolation problem (Harris and Lubensky 1981, Lubensky and McKane 1981 and references therein), and Harris and Lubensky have demonstrated the nature of the crossover between cluster distribution functions in the percolation problem and the animal problem. These studies collectively provide substantial backing for Domb's asymptotic form.

Assuming Domb's form, the next section comprises an analysis of Redelmeier's fixed animal data, while the last section contains a discussion of the results obtained. For completeness, we list the five new coefficients a_{19}–a_{24}. They are 22 964 779 660, 88 983 512 783, 345 532 572 678, 1344 372 335 524, 5239 988 770 268.

2. Analysis of series

Following the earlier procedure of Guttmann and Gaunt (1978), we have fitted successive quintuplets of coefficients a_{n-4}, a_{n-3}, a_{n-2}, a_{n-1}, a_n to the functional form $a_n = A\lambda^n n^{-\tau} \exp(-Fn^{1-\theta})$, which gives sequences of estimates of the five unknowns A, λ, τ, F and θ. By straightforward algebraic manipulation, the resulting five nonlinear equations obtained at each order can be arranged to give a nonlinear equation of a single variable (θ), which is readily found by Newton's method. The remaining parameters are then obtained by back substitution.

In table 1 we show the last 11 estimates of the five parameters for the case of fixed polyominoes (site animals). It can be seen that only for $n \geqslant 19$ do the sequences 'settle down' to regular behaviour, so the additional terms obtained by Redelmeier

Table 1. Results of a five-parameter fit to the square lattice site animals.

n	τ	λ	A	F	θ
14	0.9002	4.041 64	0.2486	0.000	−1.866
15	0.7838	4.079 46	0.2917	0.210	0.607
16	1.0530	4.063 83	0.4287	0.649	1.421
17	1.2141	4.065 92	0.1967	2.149	1.185
18	1.1883	4.065 75	0.1441	1.839	1.203
19	1.0432	4.063 80	0.3998	0.591	1.473
20	1.0209	4.063 18	0.3526	0 487	1.598
21	1.0185	4.063 10	0.3481	0.478	1.617
22	1.0134	4.062 94	0.3392	0.462	1.660
23	1.0092	4.062 81	0.3323	0.452	1.703
24	1.0075	4.062 75	0.3294	0.449	1.723

are quite invaluable for estimating the critical parameters. Linear, quadratic and logarithmic extrapolation was employed, and collectively allow us to estimate $\tau = 1.00 \pm 0.02$, $\lambda = 4.0625 \pm 0.001$, $A = 0.32 \pm 0.02$, $F = 0.43 \pm 0.04$, $\theta = 1.80 \pm 0.15$. These estimates are significantly more precise than those obtained by Guttmann and Gaunt (1978) with shorter series, the width of the confidence limits having been reduced by a factor of about five, while unbiased estimates of A and F have been made for the first time. In particular, the assumption that $\tau = 1$ made by Guttmann and Gaunt is seen to be particularly well supported.

If we now fix $\tau = 1.0$, a result held to be exact (Parisi and Sourlas 1981), we can fit successive quadruplets of coefficients to the assumed form $a_n = A\lambda^n n^{-1} \exp(-Fn^{1-\theta})$. Again, simple algebra allows us to recast the equations into an equation which is nonlinear in a single variable (θ). The results are shown in table 2, where it can be seen that good convergence is only achieved for $n > 17$. From these sequences, we obtain the biased estimates $\lambda = 4.0626 \pm 0.0002$, $A = 0.317 \pm 0.003$, $F = 0.465 \pm 0.02$ and $\theta = 1.87 \pm 0.06$. These are in agreement with the earlier biased estimates of Guttmann and Gaunt, but display a level of precision of between three and ten times greater. In particular, it appears that $\theta = 1\frac{7}{8}$ is a useful mnemonic.

Table 2. Result of a four-parameter fit to the square lattice site animals (τ assumed to be 1).

n	λ	A	F	θ
14	4.061 37	0.325 44	0.3900	1.654
15	4.061 86	0.322 32	0 4013	1 711
16	4.061 98	0.321 60	0.4055	1.727
17	4.062 15	0.320 45	0.4143	1.754
18	4.062 30	0.319 45	0.4246	1.781
19	4.062 37	0.318 96	0.4309	1.796
20	4.062 41	0.318 68	0.4351	1.805
21	4.062 44	0.318 44	0.4393	1.813
22	4.062 47	0.318 27	0.4427	1.820
23	4.062 48	0.318 15	0.4454	1.824
24	4.062 49	0.318 06	0.4477	1.828

3. Discussion

The excellent convergence that has been observed in the foregoing extrapolations lends considerable support to Domb's proposed asymptotic form. This double exponential form is particularly sensitive to small errors in coefficients, and to numerical rounding, so the convergence observed also establishes the essential correctness of the coefficients. The sensitivity of the asymptotic form assumed can perhaps best be seen by considering the free polyominoes. As we have mentioned, the expression $a_n \sim 8p_n$ deviates from equality by less than one part in 10^6 for $n = 24$, yet the free polyominoes are totally unextrapolable under these same assumptions. The reason appears to be related to the lattice structure of the loose-packed lattice used, the effects of the oscillations characteristic of loose-packed lattices being apparently sufficient to mask the very weak exponential factor $\exp(-Fn^{1-\theta})$ in the case of free polyominoes.

Recently, considerably attention has been given to the possible presence of confluent logarithmic terms in the singularity structure of a number of percolation series (see e.g. Adler and Privman 1981). Attempts to determine such confluent terms have not been particularly successful. The results obtained here, together with the demonstrated connection between lattice animals and percolation clusters (Harris and Lubensky 1981), suggest that perhaps attention should be paid to the possibility of less common exponential-type confluent terms in the case of percolation functions.

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